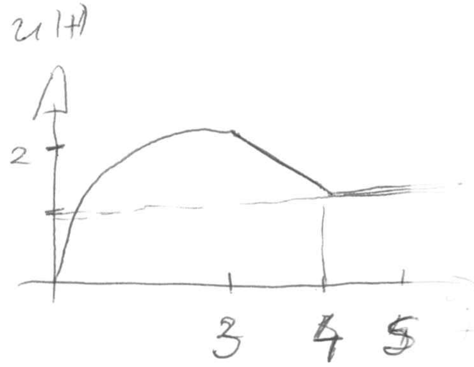
(1)

$$u(t) = \begin{cases} 2 - 2e^{-2.5t}, & 0 \le t < 3 \\ 5 - t, & 3 \le t < 4 \\ 1, & t \geq 4 \end{cases}$$



$$u(t) = \left(2 - 2e^{-2.5t}\right) H(t) - (t-3) H(t-3) + \underbrace{(t-4) H(t-}_{!}$$

Heaviside

$$U(s) = \mathcal{L}\{u(t)\} = 2\left(\frac{1}{s} - \frac{1}{s+2.5}\right) - \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^2}$$

$$= \frac{5}{s(s+2.5)} - \frac{e^{-3s} - e^{-4s}}{s^2} = \frac{5}{s(s+2.5)} - \frac{e^{-3s}(1-e}{s^2}$$

$$G(s) = -\frac{s-0.5}{s+1} \Rightarrow Y(s) = G(s)U(s)$$

$$Y(s) = -\underbrace{\frac{5(s-0.5)}{s(s+1)(s+2.5)}} + \underbrace{\frac{s-0.5}{s^2(s+1)}} e^{-3s}(1-e$$

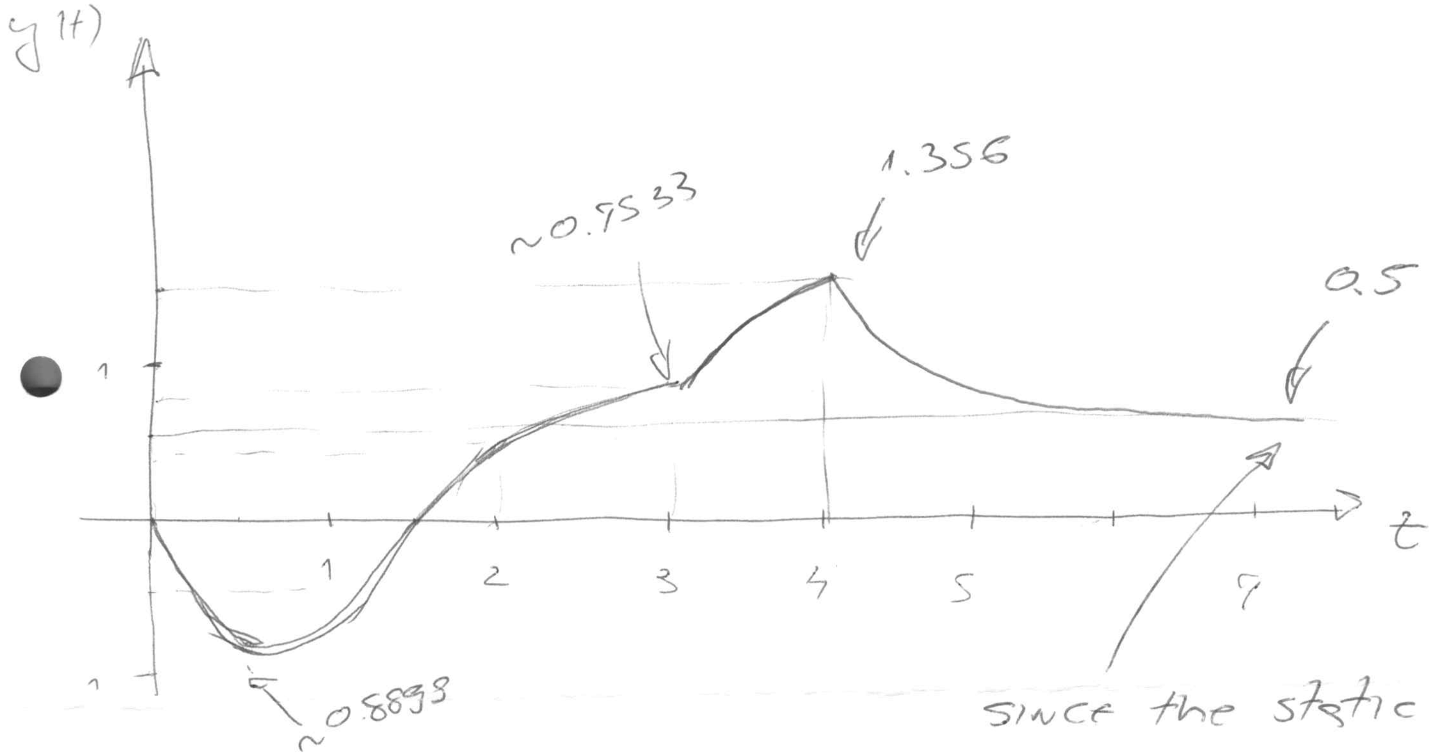
$$\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2.5} = \frac{1}{s} - \frac{5}{s+1} + \frac{4}{s+2.5}$$

$$\frac{E}{s} + \frac{F}{s^2} + \frac{G}{s+1} = \frac{1.5}{s} - \frac{0.5}{s^2} - \frac{1.5}{s+1}$$

$$y(t) = \left(1 - 5e^{-t} + 4e^{-2.5t}\right) H(t)$$
$$+ 0.5\left(3 - (t-3) - 3e^{-(t-3)}\right) H(t-3)$$
$$- 0.5\left(3 - (t-4) - 3e^{-(t-4)}\right) H(t-4)$$

To sketch the responses, we observe first that only the first term in $y(t)$ contributes in the first three seconds of the response, i.e.

$$y(t) = 1 - 5e^{-t} + 4e^{-2.5t}, \quad 0 \le t < 3$$



since the static gain is

$$K_S = \lim_{s \to 0} -\frac{s-0.5}{s+1} = 0.5$$

then for $u(\infty) = 1$

$y(\infty)$ indeed goes to 0.5

(2) $\ddot{y}(t) + 4\dot{y}(t) + 6.25\, y(t) = u(t)$

$y(0) = 1 \; ; \; \dot{y}(0) = -2$

$u(t) = H(t)$

a) $s^2 Y(s) - s y(0) - \dot{y}(0) + 4 s Y(s) - 4 y(0) + 6.25 Y(s) = U($

$$Y(s)\left(s^2 + 4s + 6.25\right) = U(s) + s + 2$$

$$Y(s) = \underbrace{\frac{1}{s^2+4s+6.25}}_{\text{forced response}} U(s) + \underbrace{\frac{s+2}{s^2+4s+6.25}}_{\text{free response}}$$

(slide 34, Laplace transform)

$$G(s) = \frac{1}{s^2+4s+6.25} \rightarrow 2^{nd}\text{-order system}$$

b) $K = \lim_{s \to 0} G(s) = \frac{1}{6.25}$ static gain

$\omega_0 = \sqrt{6.25} \rightarrow \omega_0 = 2.5$ natural frequency

$\xi = \frac{4}{2\omega_0} = 0.8 \rightarrow$ damping factor

c) <u>The response to initial conditions</u> is the free response (see above), i.e. when $U(s) \equiv 0$, therefore

$$y_{initial}(t) = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4s+6.25}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(1.5)^2}\right\} \rightarrow e^{-2t}\cos(1.5t)$$

Impulse response

$U(s)=1$

$$Y(s) = \frac{1}{(s+2)^2+(1.5)^2}\cdot 1 = \frac{1.5}{(s+2)^2+(1.5)^2}\cdot\frac{1}{1.5}$$

$$y(t)=\mathcal{L}^{-1}\{Y(s)\} = \frac{2}{3}e^{-2t}\sin(1.5t)$$

This is an underdamped system, therefore for $0\leq\xi\leq 1$ we have conjugate-complex poles and the general step response of this system is: $\left(G(s)=\frac{K\omega_0^2}{s^2+2\xi\omega_0 s+\omega_0^2}\right)$

$$* \quad y(t)=K\left\{1-e^{-\xi\omega_0 t}\left(\cos(\omega_0\sqrt{1-\xi^2}\,t)+\frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_0\sqrt{1-\xi^2}\,t)\right)\right\}$$

Since we have the value at t_2 is the maximum of the response, we have to find dy/dt, i.e.,

$$** \quad \frac{dy}{dt} = \frac{K e^{-\xi\omega_0 t}\,\omega_0}{\sqrt{1-\xi^2}}\sin(\omega_0\sqrt{1-\xi^2}\,t)$$

$$\frac{dy}{dt} = \emptyset \quad \text{for} \quad \omega_0\sqrt{1-\xi^2}\,t = K\pi \qquad K=0,\pm1,\pm2\ldots$$

so, the first maximum, $K=1$, we get

$$t_2 = \frac{\pi}{\omega_0\sqrt{1-\xi^2}} \quad \text{and} \quad *$$

we get

$$\max = y(t_2) = K\left(1+e^{-\frac{\xi}{\sqrt{1-\xi^2}}\pi}\right)$$

Here, we have $y_{MAX} = 1.56898$

and $K = 1.5$ (from the graph)

So,

$$\frac{y(t_2)}{K} - 1 = e^{-\frac{\xi}{\sqrt{1-\xi^2}}\pi}$$

$$\ln\left(\frac{y(t_2)}{k} - 1\right) = -\frac{\xi}{\sqrt{1-\xi^2}}\pi$$

$$\boxed{\xi = \sqrt{\frac{\ln\left(\frac{y(t_2)}{k} - 1\right)^2}{\pi^2 + \ln\left(\frac{y(t_2)}{k} - 1\right)^2}} = 0.7}$$

The second point in the graph, at $t = t_1$ is the inflection point, therefore we need to compute $\frac{d^2y}{dt^2} = 0$, so from **

we have

$$\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{k\, e^{-\xi\omega_0 t}\, \omega_0}{\sqrt{1-\xi^2}} \sin(\omega_0\sqrt{1-\xi^2}\, t\right.$$

$$\Rightarrow \frac{d^2y}{dt^2} = K\omega_0^2 e^{-\xi\omega_0 t}\left[\cos(\omega_0\sqrt{1-\xi^2}\, t - \frac{\xi}{\sqrt{1-\xi^2}} \sin\left(\sqrt{1-\xi^2}\,\omega_0 t\right)\right].$$

$\frac{d^2y}{dt^2} = 0$ when

*** $$\cos \omega_0\sqrt{1-\xi^2}\, t = \frac{\xi}{\sqrt{1-\xi^2}} \sin \omega_0\sqrt{1-\xi^2}\, t$$

therefore at $t_1 = 0.2227s$ from ***

we obtain

$$\frac{\sqrt{1-\xi^2}}{\xi} = tg\left(\omega_0 \sqrt{1-\xi^2}\, t_1\right)$$

$$\boxed{\omega_0 = \frac{\operatorname{arctg} \frac{\sqrt{1-\xi^2}}{\xi}}{\sqrt{1-\xi^2}\, t_1} = 5}$$